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13. The three rectangles $PQ_1R_1S_1$ (12), PQP_2Q_1 (8, 9), and $QP_2S_2R_2$ (12) show that the points R_1, Q_1, P_2, S_2 are collinear, as well as the points S_1, P, Q, R_2, and the two lines are parallel. Similar considerations may be applied to the three rectangles PSP_4S_1 (8, 9), $PQRS$ (11), QRQ_3R_2 (8, 9), and again to the series $SP_4Q_4R_4$ (12), SRS_3R_4 (8, 9), $RS_3P_3Q_3$ (12). Thus we obtain the four parallel lines $R_1Q_1P_2S_2$, S_1PQR_2, P_4SRQ_3, $Q_4R_4S_3P_3$. Similarly for the four parallel lines $R_1S_1P_4Q_4$, Q_1PSR_4, P_2QRS_3, $S_2R_2Q_3P_3$. Consequently: *The sixteen I-centers of the four triangles determined by the vertices of an inscribed quadrilateral taken three at a time, lie by groups of four on eight straight lines. The eight lines consist of two sets of four parallel lines, and the lines of one set are perpendicular to the lines of the other.*

NOTE ON CONTINUOUS FUNCTIONS.

By CAPTAIN K. P. WILLIAMS.

1. The class of continuous functions is at the same time one of the simplest and most important in analysis. On account of the character of physical phenomena, quantities whose variation is of a continuous sort are naturally the first considered. In the more recently developed topics in analysis other classes of functions are considered, and the idea of continuity does not play such a fundamental rôle. The considerations that follow are intended primarily for those who have just become familiar with the properties of continuous functions as developed from a rigorous point of view. As they are of a simple nature they may serve as an easy introduction to the study of classes of functions where one's starting point is no longer the idea of continuity.

When we examine the properties of continuous functions we find that they fall into two rather broad classifications. There are those properties that have to do with the behavior of the function in the immediate vicinity of a point, and those that relate to some character with reference to the interval of definition. We could call them properties "im kleinen," and properties "im grossen." The definition of continuity itself is an example of the first class of properties, and the theorem that every continuous function has actual extrema in a closed interval, an example of the second. Thus we see how strongly properties of a function "im kleinen" effect its behavior "im grossen." Intimately connected with any property "im kleinen" is the question of uniformity; that is we inquire whether the property in question occurs in a uniform manner throughout the interval of definition. And in an analogous manner we can ask whether a property "im grossen" is true in every sub-interval. With such ideas before us we can examine easily how far various properties of continuous functions are uniquely characteristic.

2. We consider a function $f(x)$ which approaches a definite limit at every point, but which is not necessarily continuous at any point in the interval (a, b). Denote by $f'(x)$ the limit approached. We first prove the

Lemma. *The function $f'(x)$ has a definite limit at every point.*

Let x_0 be any point on the interval (a, b), and ϵ any positive number. By hypothesis we can find η such that

$$|f(x') - f(x'')| < \epsilon,$$

when $|x_0 - x'| < \eta$, and $|x_0 - x''| < \eta$ and x' and x'' are both different from x_0. Take x_1 and x_2 any two points in the interval $|x_0 - x| < \eta$, then we can find x' and x'' so that

$$|f'(x_1) - f(x')| < \epsilon, \qquad |f'(x_2) - f(x'')| < \epsilon,$$

while x' and x'' are also both in the interval $|x_0 - x| < \eta$ and are different from x_0. It follows then that

$$|f'(x_1) - f'(x_2)| < 3\epsilon,$$

which establishes the property announced.

We can go a step farther and state

Theorem 1. *The function $f'(x)$ is continuous.*

We have

$$|f'(x_0) - f(\bar{x})| < \epsilon, \qquad \text{for} \qquad |x_0 - \bar{x}| < \delta,$$

and from what we have just proved,

$$|f'(x_1) - f'(x_2)| < \epsilon, \qquad \text{for} \qquad |x_0 - x_1| < \eta, \qquad |x_0 - x_2| < \eta.$$

Let now $\bar{\eta}$ be the smaller of η and δ, and let x_1 and x_2 be in the interval $|x - x_0| < \bar{\eta}$. We can find an $\bar{x}$ in this interval such that

$$|f'(x_1) - f(\bar{x})| < \epsilon,$$

while at the same time from the manner of choice of $\bar{\eta}$,

$$|f'(x_0) - f(\bar{x})| < \epsilon, \qquad |f'(x_1) - f'(x_2)| < \epsilon.$$

We thus have

$$|f'(x_0) - f'(x_2)| < 3\epsilon \qquad \text{for} \qquad |x_0 - x_2| < \bar{\eta},$$

which shows at once that $f'(x)$ is a continuous function.

Theorem 2. *If $f(x)$ approaches a limit uniformly on (a, b) it will be a continuous function.*

Under the hypothesis we can find η such that

$$|f'(x_0) - f(x)| < \epsilon, \qquad \text{for} \qquad |x_0 - x| < \eta,$$

uniformly for x_0 on (a, b), the number ϵ being arbitrary. Take x_1, x_2 any two points in $|x - x_0| < \eta/2$; then

$$|f(x_2) - f'(x_0)| < \epsilon.$$

But we also see that

$$|f(x_2) - f'(x_1)| < \epsilon, \qquad |f(x_0) - f'(x_1)| < \epsilon.$$

It therefore follows that

$$| f(x_0) - f'(x_0) | < 3\epsilon,$$

so that

$$f(x_0) = f'(x_0). \qquad \text{Q. E. D.}$$

Thus in place of the ordinary definition of continuity we could adopt the following: *A continuous function is one that possesses a limit at every point, and in which the limit is approached uniformly.* This gives us a property "im kleinen," if existent uniformly, as completely characteristic of continuity.

3. We now put $\delta(x) = f'(x) - f(x)$ and examine the nature of the function thus defined. Let δ be the greatest lower bound of the function $| \delta(x) |$ on (a, b). We prove

Theorem 3. *The quantity δ is equal to zero.*

Suppose $\delta \neq 0$; then at every point $| \delta(x) | > \eta > 0$. Then for any x' we can find a ξ such that

$$| f(x) - f(x') | > \frac{\eta}{2}, \qquad \text{when} \qquad | x - x' | < \xi, \qquad x \neq x'.$$

Consider now a point x_0. Since $f'(x_0)$ is the limit approached we have

$$| f'(x_0) - f(x) | < \frac{\eta}{8}, \qquad \text{when} \qquad | x - x_0 | < \xi,$$

ξ being a sufficiently small positive quantity. Therefore

$$| f(x') - f(x'') | < \frac{\eta}{4}, \qquad \text{when} \qquad | x' - x_0 | < \xi, \qquad \text{and} \qquad | x'' - x_0 | < \xi.$$

Now consider x' as fixed. We can then find an interval in the interval $| x - x_0 | < \xi$, containing x', and such that when x'' is in this interval we also have

$$| f(x') - f(x'') | > \frac{\eta}{2},$$

and have thus reached a contradiction.

Corollary. *In every sub-interval of (a, b) the greatest lower bound of $\delta(x)$ is zero.*

Denote by $\bar{x}$ a point such that in every vicinity of it the greatest lower bound of $| \delta(x) |$ is zero. It follows that

Theorem 4. *The points $\bar{x}$ are everywhere dense.*

4. We next consider a property "im grossen," which in itself is not sufficient to completely characterize continuity,[1] but which we shall suppose is true in every sub-interval.

Theorem 5. *Let $f(x)$ be a function such that in every interval containing x_0 it assumes all values between any two of its values; further suppose it assumes any value only once; then $f(x)$ is continuous at $x = x_0$.*

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[1] K. P. Williams, Concerning a Certain Totally Discontinuous Function, *Bulletin of the American Mathematical Society*, vol. 21, 1914, pp. 117–120.

Let (a_1, b_1) be any interval containing x_0, and let M_1, m_1 be the bounds of the function on (a_1, b_1); then the function assumes all values between m_1 and M_1. Take the case where $m_1 < f(x_0) < M_1$. Continue in this way where a_1, a_2, a_3, $\cdots$ approach x_0, and likewise b_1, b_2, b_3, $\cdots$. Let M_i, m_i be the bounds of the function on (a_i, b_i). Then both M_1, M_2, $\cdots$ and m_1, m_2, $\cdots$ approach $f(x_0)$. Suppose for instance that M_1, M_2, M_3, $\cdots$ do not approach $f(x_0)$, but have $M > f(x_0)$ as greatest lower bound. Take k, where $f(x_0) < k < M$, then there is one value $\bar{x}$ on (a_1, b_1) such that $f(\bar{x}) = k$. Let i be large enough that (a_i, b_i) does not contain $\bar{x}$; then we must have $M_i < k$, or otherwise there would be a root of $f(x) = k$ on (a_i, b_i), which is impossible. But this contradicts the assumption that M is the greatest lower bound of M_1, M_2, $\cdots$. Since M_1, M_2, $\cdots$ and m_1, m_2, $\cdots$ both tend to $f(x_0)$ it follows that $f(x)$ is continuous at $x = x_0$.

The cases other than that where $m_1 < f(x_0) < M_1$ follow in a similar way.

Corollary. *The theorem is true if $f(x)$ assumes any value at most n times, n a constant.*

NOTE ON FUNCTIONS WHICH APPROACH A LIMIT AT EVERY POINT OF AN INTERVAL.

By E. W. CHITTENDEN, University of Illinois.

In the foregoing paper Captain Williams has discussed properties of functions which approach a limit at every point of an interval. It is the purpose of this note to present the following theorem:

THEOREM. *If a function $f(x)$ has a limit $f'(x)$ at every point x of an interval (a, b), then for every positive number σ, however small, the number of points at which the measure of discontinuity (saltus) exceeds σ is finite, and the set of points at which $f(x)$ differs from the continuous function $f'(x)$ is at most enumerably infinite.*

At any point x of the interval (a, b) there is for any small positive number e an open interval (segment) $S_{xe} = (x - h < x' < x + h)$ such that for any point x' in the segment, distinct from x,

$$|f(x') - f'(x)| < \frac{e}{4},$$

and also, since $f'(x)$ is continuous,

$$|f'(x') - f'(x)| < \frac{e}{4}.$$

Hence

$$|\delta(x')| = |f'(x') - f(x')| < \frac{e}{2}.$$

Therefore the oscillation of the function $\delta(x)$ on the set obtained from S_x by omitting the point x is less than e.

Every point of the interval (a, b) is enclosed in a segment S_x except the points a and b, for which $S_a = (a \leqq x < a + h)$, $S_b = (b - h < x \leqq b)$. (It is to be